Stability of spinning ring solitons of the cubic-quintic nonlinear Schrödinger equation

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Abstract

We investigate stability of (2+1)-dimensional ring solitons of the nonlinear Schrödinger equation with focusing cubic and defocusing quintic nonlinearities. Computing eigenvalues of the linearised equation, we show that rings with spin (topological charge) $s = 1$ and $s = 2$ are linearly stable, provided that they are very broad. The stability regions occupy, respectively, 9% and 8% of the corresponding existence regions. These results finally resolve a controversial stability issue for this class of models.

Key words: solitons, ring, nonlinear Schrödinger equation
PACS: 42.65.Tg

1 Introduction

Recently, much interest has been focused on *spinning* optical solitons, i.e., those carrying topological charge, in both (2+1)D [1–5] and (3+1)D [7–10] geometries. A spinning soliton has an embedded phase dislocation and carries intrinsic angular momentum. The integer number of phase rotations around the dislocation is the soliton's topological charge or "spin".

Broadly speaking, spinning solitons can be divided into two classes: (i) dark, i.e., vortices produced by a phase dislocation which is embedded in an infinite background; and (ii) bright, with the vortex core embedded in a bright

(localised) multidimensional soliton proper; the amplitude of which vanishes at infinity. In this Letter we consider bright spinning solitons in the (2+1)D geometry. In terms of nonlinear optics, (2+1)D solitons may be naturally realised as spatial solitons in the form of cylindrical beams in a bulk medium, or, alternatively, as spatiotemporal solitons in the form of fully localised "light bullets" propagating in a planar waveguide (film). Due to the presence of the vorticity, the soliton's cross section has an annular shape. We shall refer to them in what follows below as *localised vortex solitons* (LVSs). As for the (3+1)D solitons, they are spatiotemporal "bullets" propagating in bulk media.

Models that may give rise to stable multidimensional, (2+1)D or (3+1)D, solitons must necessarily have nonlinearity which prevents dynamical collapse. Well-known examples of collapse-free nonlinearities are $\chi^{(2)}$ (second-harmonic-generating), saturable, and cubic-quintic (CQ); the cubic and quintic components being, respectively, self-focusing and self-defocusing. All these nonlinearities occur in various optical media.

However, unlike ground-state (zero-spin) (2+1)D bright solitons, for the spinning ones the absence of collapse does not guarantee dynamical stability. In fact, a LVS tends to be strongly destabilised by azimuthal perturbations which break it up into several separating zero-spin bright solitons (the latter are stable). In (2+1)D models with $\chi^{(2)}$ and saturable nonlinearities, numerical simulations had revealed strong azimuthal instability [1,2], which was later observed experimentally in a $\chi^{(2)}$ medium [3]. A similar instability of (3+1)D rings resulting from the $\chi^{(2)}$ nonlinearity has been found in Ref. [10]: a 3D LVS (in fact, it is a torus) breaks up due to an azimuthal perturbation, and the resulting zero-spin solitons fly off in directions tangential to the ring.

In terms of the LVS stability, more promising are models with competing nonlinearities. The first step in this direction was the study of (2+1)D rings in the CQ nonlinear Schrödinger equation. Simulations reported by Quiroga-Teixeiro and Michinel [4] showed that broad rings found (by means of the variational approximation and direct numerical methods) in that model not only did not demonstrate growth of small perturbations, but also survived collisions, thus appearing fairly stable. However, a similar analysis for the (3+1)D CQ model [9] has demonstrated that narrow LVSs were broken up quickly, while broad rings were destabilised much slower by azimuthal perturbations. However, eventually all the LVSs for which a definite numerical result could be obtained were unstable. This suggests re-checking the above-mentioned stability of LVSs in the 2D model reported in [4]. Rerunning simulations for the same cases which were considered in [4], it was established [6] that, in fact, they are *also subject* to the weak instability against azimuthal perturbations, provided that the simulations are long enough.

As 2D and 3D LVSs in the CQ model become very broad, it still remains

unknown whether the growth rate of their instability against azimuthal perturbations gradually vanishes in the limit of infinitely broad rings, or if there is a clearly defined transition to truly stable solitons. The problem of discerning between very weak instability and true stability may be of little importance for applications, as experiments are always carried out in finite samples, which do not have enough room for the development of an instability if it is extremely weak. Nevertheless, the issue is of principal interest and is therefore worthy of consideration.

Another model with competing nonlinearities which may be promising for the generation of stable rings combines the $\chi^{(2)}$ and *self-defocusing* cubic $[\chi^{(3)}]$ nonlinearities. It is necessary to say that no conventional nonlinear material with strong $\chi^{(2)}$ nonlinearity directly satisfies the requirement of this model to have a negative $\chi^{(3)}$ coefficient at both the fundamental- and second-harmonic frequencies (see below). However, two possibilities to create the necessary *effective* $\chi^{(3)}$ nonlinearity have been proposed: (i) by creating a layered medium in which layers providing for the $\chi^{(2)}$ nonlinearity periodically alternate with others that account for the self-defocusing Kerr nonlinearity, and (ii) by engineering special $\chi^{(2)}$ quasi-phase-matched gratings [12]. In the latter case, induced $\chi^{(3)}$ and intrinsic $\chi^{(2)}$ nonlinearities may be equal in strength, and the former one may be given either sign.

Recently, LVSs were considered in this $\chi^{(2)} - \chi^{(3)}$ model [11]. Using direct simulations and linear stability analysis, it has been shown that narrow rings demonstrate typical breakup into zero-spin bright solitons initiated by the azimuthal instability, but very broad (flat-top) LVSs with the spin (topological charge) $s = 1$ and $s = 2$ were indeed found to be *completely stable*. In these two cases, the stability region is, respectively, $\approx 8\%$ and $\approx 5\%$ of the corresponding existence domain.

In the *cascading limit*, corresponding to large wave-vector mismatch, the $\chi^{(2)} - \chi^{(3)}$ model reduces to the CQ model, which suggests that there may be a chance to find completely stable LVSs in the CQ model too, which is a subject of the present work. We will employ the same techniques that were applied in Ref. [11] to the $\chi^{(2)} - \chi^{(3)}$ model, i.e., rigourous computation of the stability eigenvalues. This will overcome limitations of "phenomenological" analyses of the CQ model carried out in previous works, which relied on simulations of perturbed solitons to directly test their stability, or simulations of the linearised equation to estimate the largest growth rate of instability. A shortcoming of both methods is that they can miss very weak instability. We will demonstrate that LVSs in the CQ model are *truly* stable if their width (or energy) exceeds a certain critical value. This result completes the investigation of the CQ model in the (2+1)D case, which was the subject of many above-mentioned works, and suggests a challenging question: can spinning "bullets" with sufficiently large energy be stable in the same model in the (3+1)D case?
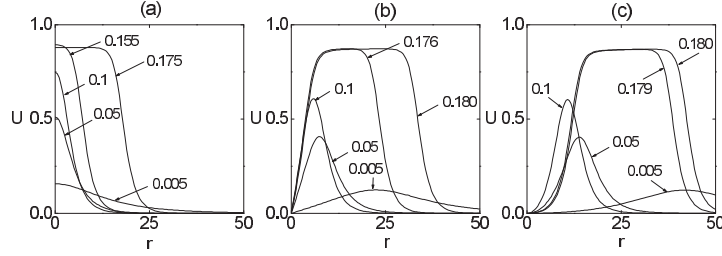
Fig. 1. The stationary (2+1)D spinning-soliton solutions: (a) $s = 0$, (b) $s = 1$, and (c) $s = 2$. The values of κ are indicated near the curves.

2 The ring solitons

Following the derivation of Ref. [8], but assuming two transverse spatial dimensions, we arrive at the dimensionless CQ equation

$$i\frac{\partial u}{\partial z} + \nabla_\perp^2 u + |u|^2\, u - |u|^4\, u = 0. \tag{1}$$

where u is the envelope of the electromagnetic wave propagating along the z-direction in the optical medium. In the case of the cylindrical beams in the bulk medium (i.e., spatial solitons, see above), the Laplacian in Eq. (1) is the diffraction operator acting on the transverse spatial coordinates x and y.

In the alternative case of spatiotemporal solitons in the planar waveguide, y is replaced by a properly scaled temporal variable, $\tau \equiv t - z/V_0$, where t is time, and V_0 is the group velocity of the carrier wave. In the latter case, the part $\partial^2/\partial\tau^2$ in the Laplacian accounts for the temporal dispersion (which must be *anomalous*, in order to have to right sign), rather than diffraction, and LVS will be, in unrescaled coordinates, a compressed (elliptic) ring moving in its own plane.

Ring solitons are localised stationary solutions to Eq. (1) of the form

$$u = U(r)\exp(is\theta + i\kappa z), \tag{2}$$

where r and θ are polar coordinates in the (x, y) plane, κ is a wave number shift (relative to the carrier wave), and the integer s is the spin. The amplitude U can be taken to be real, obeying an equation

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} - \frac{s^2}{r^2}U - \kappa U + U^3 - U^5 = 0. \tag{3}$$

We solved Eqs. (3) by means of the relaxation technique. When κ is small (a low-power regime), LVSs are narrow. The beam's amplitude at first increases

with κ and then saturates, while the ring's width keeps increasing because of the self-defocusing effect of the cubic term.

The wave number κ parameterises a family of stationary solutions, examples of which are displayed in Fig. 1. The existence regions for solitons in the 2D and 3D versions of the CQ model are known [4,8]:

$$0 < \kappa < \kappa_{\text{offset}}^{(2D)} \approx 0.18; 0 < \kappa < \kappa_{\text{offset}}^{(3D)} \approx 0.15, \qquad (4)$$

and in both cases they practically do not depend on the soliton's spin [8]. The width and energy of LVS diverge as $\kappa \to \kappa_{\text{offset}}$. Note that a soliton solution to the 1D version of Eq. (1) is known in an exact elementary form, the corresponding exact offset wave number being $\kappa_{\text{offset}}^{(1D)} = 3/16 \equiv 0.1875$, so that the values (4) are close to it.

3 Stability

As a precursor to the full linear stability analysis of the ring solitons, we first consider the modulational stability of plane continuous-wave (cw) solutions to Eq. (1). This may give some insight into the stability of broad rings as they tend to these cw solutions in the limit $\kappa \to \kappa_{\text{offset}}$. The cw solutions with the propagation constant κ are

$$u_0 = a_0 \exp(i\kappa z), \; a_0^2 = \frac{1}{2}\left(1 \pm \sqrt{1 - 4\kappa}\right). \qquad (5)$$

We take small perturbations to the plane wave of the form

$$u_1 = [\alpha \exp\left(ikx + i\Omega z\right) + \beta \exp\left(-ikx - i\Omega^* z\right)] \exp(i\kappa z), \qquad (6)$$

where k is an arbitrary perturbation wave number, and Ω is the stability eigenvalue (the asterisk stands for the complex conjugation); the cw solutions being stable if Ω is real for all real k. Substituting into Eq. (1) the perturbed solution $u = u_0 + u_1$ and linearising around u_0, we find after some straightforward algebra that:

$$\Omega^2 = k^2 + k^4 \pm k^2\sqrt{1 - 4\kappa} - 4k^2\kappa. \qquad (7)$$

It is easy to demonstrate that for the cw branch with the upper sign in Eq. (5) (i.e., with the larger amplitude), $\Omega^2 \geq 0$ for all k , hence, this branch is always stable, while the other one is not. As it was said above, the LVS solutions
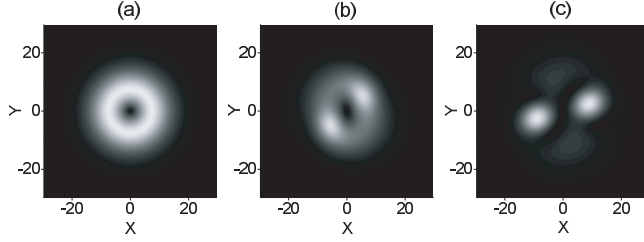
Fig. 2. Gray-scale contour plots illustrating the evolution of an unstable ring soliton with $s = 1$ and $\kappa = 0.05$: (a) $z = 0$; (b) $z = 310$; (c) $z = 330$.

of the CQ model that we are dealing with tend to the stable cw solution as $\kappa \to \kappa_{\text{offset}}$, giving an initial indication that, as the rings broaden, they may become stable.

This conclusion is suggested by simulations reported in Ref. [4], where sufficiently broad rings appeared to be stable, whereas narrower ones were definitely unstable against azimuthal perturbations (see Fig. 2). On the other hand, as was mentioned above, more accurate (longer) simulations of the cases that were considered in that work show that weak instability still occurs. To perform the direct simulations we used the Crank-Nicholson scheme as a finite-difference approximation to propagation equations (1). The corresponding system of nonlinear equations was solved by means of the Picard iteration method (see details in Ref. [13]). Typically, we chose equal transverse grid sizes $\Delta x = \Delta y = 0.4$ and the longitudinal grid size was $\Delta z = 0.02$. Thus, direct simulations show a general trend of suppression of the azimuthal instability with the increase of the size of LVSs. This may be sufficient to expect experimental observability of LVSs, but, following this analysis, one cannot predict if the instability may be completely eliminated, provided that the size of LVS exceeds some critical value.

The stability issue can only be resolved in the precise sense by comprehensive analysis of eigenmodes of small perturbations around LVS. To this end, we add infinitesimal complex perturbations $\epsilon(z, r, \theta)$ to the stationary solutions of Eqs. (1) and (3) with the vorticity s, cf. Eq. (6),

$$u = [U(r) + \epsilon(z, r, \theta)] \exp(is\theta + i\kappa z). \tag{8}$$

A general perturbation $\epsilon(z, r, \theta)$ may always be expanded into a series, with each term having its own vorticity J, so that a generic independent perturbation term is

$$\epsilon = \xi_J^+(r) \exp\left(i\lambda z + iJ\theta\right) + \xi_J^-(r) \exp\left(-i\lambda^* z - J\theta\right), \tag{9}$$

6

where λ is the (generally complex) LVS's instability eigenvalue. Substituting this into Eqs. (1) and linearising, we arrive at a non-self-adjoint eigenvalue problem:

$$\lambda \vec{\xi}_J = \begin{bmatrix} C_+ & D \\ -D & -C_- \end{bmatrix} \vec{\xi}_J, \tag{10}$$

where $\hat{\xi}_J \equiv (\xi_J^+, \xi_J^-)$, $C_\pm = \hat{L}_J^\pm + 2U^2 - 3U^4$, $D = U^2 - 2U^4$, and

$$\hat{L}_J^\pm \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}(s \pm J)^2 - \kappa.$$

Instability is accounted for by eigenvalues with $\mathrm{Im}\lambda \neq 0$ (the present system being Hamiltonian, eigenvalues appear in complex conjugate pairs or quadruplets). The continuous spectrum of the eigenvalues consists of real intervals $\kappa \leq \lambda < \infty$ and $-\infty < \lambda \leq -\kappa$.

To analyse the eigenvalue problem (10), we replaced the differential operators by their fifth-order finite-difference approximations and solved the resulting algebraic eigenvalue problem numerically. We mostly used grids with 400 to 800 points, but up to 1200 points were used in regions where a change of the stability occurs. To verify the precision of the numerical code, we also used another technique based on the relaxation method for solving two-point boundary-value problems. Although limited to finding real eigenvalues, the latter method admits a high degree of precision control without much of the computational overhead of other methods. For instance, it has been recently used to a great effect in finding a small stability window for higher-order solitons in a third-harmonic-generation model, which would have otherwise been overlooked [14]. Comparison between the spectral and relaxation methods has shown that the former one has good precision for the number of the grid points used: the numerical error in calculating the stability-boundary values of κ is estimated to be $\delta\kappa_{\mathrm{st}} \sim 10^{-5}$ for 1200 grid points.

Results of the linear stability analysis for the fundamental ($s = 1$) and next-order ($s = 2$) LVSs are summarised in Figs. 3 and 4. We considered perturbations with $J = 0, ..., \pm 5$, and have found that instability of the fundamental LVSs is not generated by perturbations with $J > 3$. The most persistent instability mode in the case $s = 1$ corresponds to $J = \pm 2$. Subsequent direct simulations demonstrate that this instability mode, if it takes place, initiates eventual breakup of the ring into several zero-spin solitons. In Ref. [2] an estimate for the number of filaments N resulting from the destruction of the LVS is given as $N \approx 2s$. Our results agree with this estimate: for LVSs with $s = 1$, dominant instability corresponds to $J = \pm 2$ and a singly charged LVS breaks
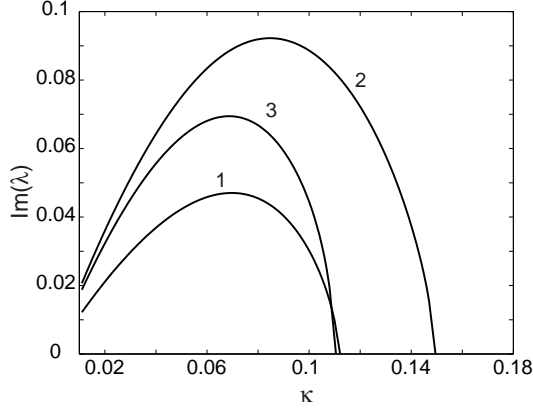
Fig. 3. Unstable eigenvalues for the ring solitons with $s = 1$. The spin J of the azimuthal perturbation is indicated next to each curve (the perturbations with $J > 3$ caused no instability, therefore they are not displayed). Only Im(λ) is shown. Note that the dominant instability has $J = 2$, and it vanishes at $\kappa \approx 0.16$, while the ring solitons exist up to $\kappa = \kappa_{\mathrm{offset}} \approx 0.18$.
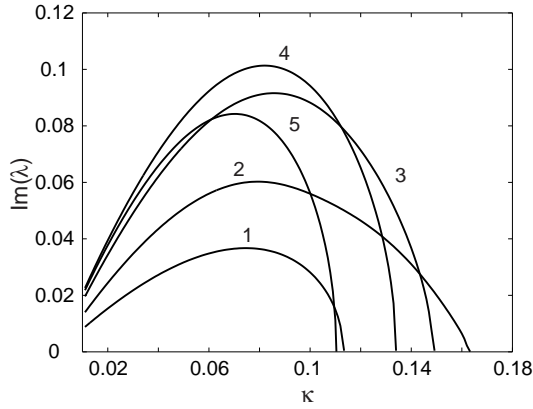

Fig. 4. The same as in the previous figure but for $s = 2$.

into two filaments, as can be seen in Fig. 2. (This agreement largely holds for the $s = 2$ case as well where for most of the unstable domain the dominant instability corresponds to $J = \pm 4$ and typically doubly charged LVSs break into four filaments.)

In all the cases considered, we have found that there is a *stability-change* value κ_{st}, at which the largest instability eigenvalue Imλ vanishes, and remains, along with all the other ones, exactly (up to the numerical accuracy) equal to zero in the *stability window*, $\kappa_{\mathrm{st}} < \kappa < \kappa_{\mathrm{offset}}$ (recall κ_{offset} is the upper existence boundary of the LVS family); in other words, thin LVSs are unstable and broad ones are stable. The existence of the window is clearly illustrated by Fig. 3. For the fundamental LVSs ($s = 1$), the stability window occupies $\approx 9\%$ of the existence domain $[0, \kappa_{\mathrm{offset}}]$.

For the LVSs with $s = 2$, a similar situation takes place. The dominant instability in a larger part of their existence domain is generated by the perturbation with $J = 4$, but for broad LVSs it is overtaken by the $J = 2$ mode. The linear spectrum of the LVSs with $s = 2$ contains a stability window which occupies $\approx 8\%$ of the existence region.

It appears that stable LVSs cannot have the value of the spin larger than 2. In particular, the LVSs with $s = 3$ were found to demonstrate a persistent weak instability associated with the $J = \pm 1$ perturbation modes at all the values of κ, and it seems very plausible that higher-order LVSs will continue to do so.

In the work of Quiroga-Teixeiro and Michinel [4], the authors use variational techniques to make an estimate for the width of the stability window. Using the formulas derived in Ref. [4] the ratio of $\kappa_{\text{st}}/\kappa_{\text{offset}}$ for $s = 1$ and $s = 2$ LVSs is 0.803 and 0.838 respectively. In other words according to the variational analysis of [4] the stability window occupies approximately 20% and 16% of the existence domain $[0, \kappa_{\text{offset}}]$ for singly and doubly charged LVS. These estimates correctly predict the existence of stability domains for both $s = 1$ and $s = 2$ LVSs and also the decrease in stability window size for the doubly charged LVSs relative to the singly charged. The factor of 2 difference between the estimates and the numerical results obtained in this work is quite reasonable considering the ansatz used in Ref. [4] becomes increasingly less accurate for large κ i.e. where the stable LVSs exist.

Lastly, it is relevant to mention that, in the work [6], a variational approach was used to study, in an analytical form, instability of LVSs in the (2+1)D and (3+1)D versions of the CQ model against a special perturbation mode in the form of an infinitesimal shift in the position of the vortex core relative to the broad soliton as a whole. In terms of the expansion (9), this mode corresponds to $J = \pm 1$. If δH is a small variation of the value of the soliton's Hamiltonian generated by the infinitesimal off-centre shift of the vortex core, an instability condition which is well-known from the general soliton stability theory is $\delta H < 0$ [15]. Assuming κ close enough to κ_{offset}, which is necessary to have broad solitons whose outer radius is much larger than the radius of the vortex core, it has been shown that a shift of the core leads to a *decrease* of an effective Hamiltonian of the interaction between the core and the outer rim of the soliton, which implies an instability. Because the interaction Hamiltonian is exponentially small in the case of the broad LVS, the instability is also expected to be exponentially weak. This instability mode was predicted for $s = 1$ and 2, but not for $s \geq 3$. It is noteworthy too that only for $s = 1$ the thus predicted instability is linear (exponentially growing), while for $s = 2$ it is nonlinear (*sub*exponential).

From our numerical results, we have indeed found a weak instability mode with $J = 1$ for the $s = 1$ and $s = 2$ rings, see Fig. 5. To the limit of our numerical
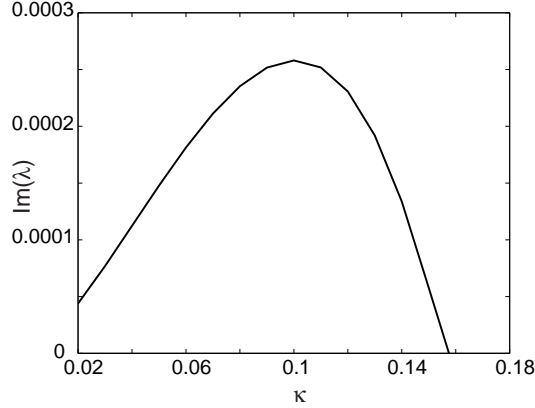
Fig. 5. Weak instability of the ring solitons with $s = 1$ resulting from $J = \pm 1$ perturbations. To the limit of the numerical method, this mode has zero instability growth rate at $\kappa \geq 0.16$. A similar mode can be found for the $s = 2$ rings.

accuracy, the corresponding instability growth rate $\mathrm{Im}\lambda$ vanishes at $\kappa \approx 0.16$, and then remains equal to zero up to the point $\kappa = \kappa_{\mathrm{offset}}^{(2D)} \approx 0.18$, at which the size of the ring becomes infinitely large, see Eq. (4). It is not completely clear yet whether an extremely small exponentially vanishing unstable eigenvalue exists past the value $\kappa \approx 0.16$, but the issue is purely formal, the broad rings being stable in any practical sense.

In conclusion, we have found that localised vortex solitons, with values of the spin 1 and 2, of the two-dimensional cubic-quintic nonlinear Schrödinger equation are linearly stable in a finite interval of the propagation constant corresponding to broad solitons with large power. These results make an important step forward in the resolution of the controversial stability issue in this class of models.

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